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Stable homology through scanning
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Lecture 2, October 11

Classifying spaces of groups

Goal of next two lectures (and rest of course):

$$\left.\begin{array}{l}\text{recognize when } G \simeq \Omega BG \\ \quad\text{and when } X \simeq BG\end{array}\right\} \Rightarrow G \simeq \Omega X.$$

Definition: if G is a topological group, its classifying space BG is the space defined up to homotopy equivalence by the existence of a principal G-bundle

$$\begin{array}{c} G \to EG \\ \downarrow \\ BG \end{array}$$

with EG contractible.

(principal G-bundle means $G \curvearrowright EG$ freely s.t. each fiber is one orbit, and local trivializations look like $G \curvearrowright G \times \mathcal{U} \to \mathcal{U}$)

Why is BG called a "classifying space"? Because there is a bijection:

$$\left\{\begin{array}{c}G\text{-bundles over } X \\ \text{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{maps } X \to BG \\ \text{up to homotopy}\end{array}\right\}$$

Examples of classifying spaces:

$B\mathbb{C}^* = \mathbb{C}P^\infty$ — classifies complex line bundles

$\mathbb{C}^* \curvearrowright \mathbb{C}^\infty - \{0\}$ with quotient $\mathbb{C}P^\infty$

Why is $\mathbb{C}^\infty - \{0\}$ contractible?

Take straight-line homotopies $(x_1, x_2, \dots)$
$\longrightarrow (0, x_1, x_2, \dots)$
$\longrightarrow (1, 0, 0, \dots)$

$BG = K(G,1)$ for G discrete

A principal G-bundle for discrete G is a covering map, so $\widetilde{BG} = EG$ is contractible:
$\pi_1(BG) = G$ while $\pi_n(BG) = \pi_n(EG) = 0$ for $n \geq 2$

$BGL_k\mathbb{R}$ = space of linear k-planes in $\mathbb{R}^\infty$ — classifies k-dim. vector bundles
($EGL_k\mathbb{R}$ = space of k linearly independent vectors in $\mathbb{R}^\infty$)

$B\mathrm{Homeo}(M)$ = space of subsets of $\mathbb{R}^\infty$ homeomorphic to M — classifies fiber bundles with fiber M
($E\mathrm{Homeo}(M)$ = space of embeddings $M \hookrightarrow \mathbb{R}^\infty$)

BS_n = space of n-element subsets of $\mathbb{R}^\infty$ — classifies n-sheeted covering spaces
(ES_n = space of sequences of n distinct points in $\mathbb{R}^\infty$)

Classifying maps

If M is an n-dimensional smooth manifold, its tangent bundle $TM \to M$ corresponds to some map $g: M \to BGL_n\mathbb{R}$.

Q: How can we explicitly realize this classifying map?

A: Choose a smooth embedding
$$f: M \hookrightarrow \mathbb{R}^\infty.$$

For each $p \in M$, the tangent plane at $f(p)$ — i.e. $f(T_pM)$ — is an n-plane in $\mathbb{R}^\infty$

Define $g: M \to BGL_n\mathbb{R}$ $\longleftarrow$ <u>space</u> of n-planes in $\mathbb{R}^\infty$

by $p \mapsto f(T_pM)$.

Q: If $\begin{matrix} M \to Y \\ \quad\ \downarrow \pi \\ \quad\ X \end{matrix}$ is a fiber bundle with fiber M,

what is the classifying map $g: X \longrightarrow B\mathrm{Homeo}(M)$?

A: Choose an embedding $f: Y \hookrightarrow \mathbb{R}^\infty$ of the total space.

For each $x \in X$, the image $f(\pi^{-1}(x))$ of the fiber over x is a subset of $\mathbb{R}^\infty$ homeomorphic to M.

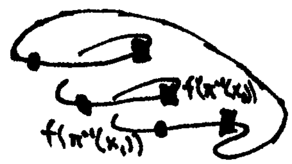
Define $g: X \longrightarrow B\mathrm{Homeo}(M)$ $\longleftarrow$ <u>space</u> of subsets of $\mathbb{R}^\infty$ homeo. to M

by $x \longmapsto f(\pi^{-1}(x))$.

Q: If $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$ is an n-sheeted covering map,

what is the classifying map $g: X \to BS_n$?

A: Choose an embedding $f: Y \hookrightarrow \mathbb{R}^\infty$



For each $x \in X$, the image $f(\pi^{-1}(x))$ of the preimage is an n-element subset of $\mathbb{R}^\infty$.

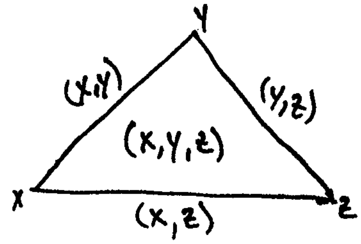
Define $g: X \longrightarrow BS_n$ $\longleftarrow$ <u>space</u> of n-element subsets of $\mathbb{R}^\infty$

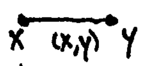
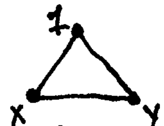
by $x \longmapsto f(\pi^{-1}(x))$

Constructing BG in general

We first take G to be a discrete group.

Let EG be the simplicial complex with n-simplices labelled by $(g_0, \ldots, g_n)$, $g_i \in G$ and the obvious face identifications:



Why is EG contractible?
Each simplex  sits inside ;
take the straight-line homotopy to 1: 

In barycentric coordinates:
$(t_0^{g_0}, \ldots, t_n^{g_n})$
$= (0^1, t_0^{g_0}, \ldots, t_n^{g_n})$
$\longrightarrow (1^1, 0^{g_0}, \ldots, 0^{g_n})$

G acts freely on EG by
$g \cdot (g_0, g_1, \ldots, g_n) = (g \cdot g_0, g \cdot g_1, \ldots, g \cdot g_n)$

BG is defined to be the quotient EG/G.
The simplices in the quotient can be uniquely labeled as $(1, g_1, \ldots, g_n)$, but the face identifications are not so nice in these coordinates

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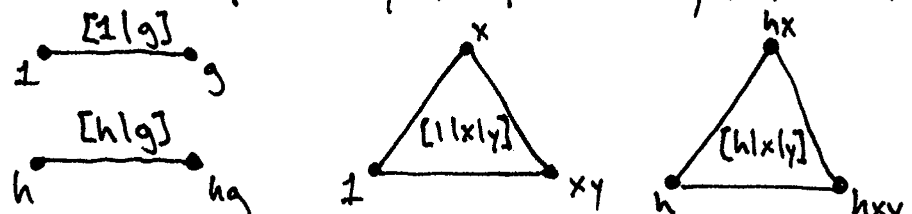
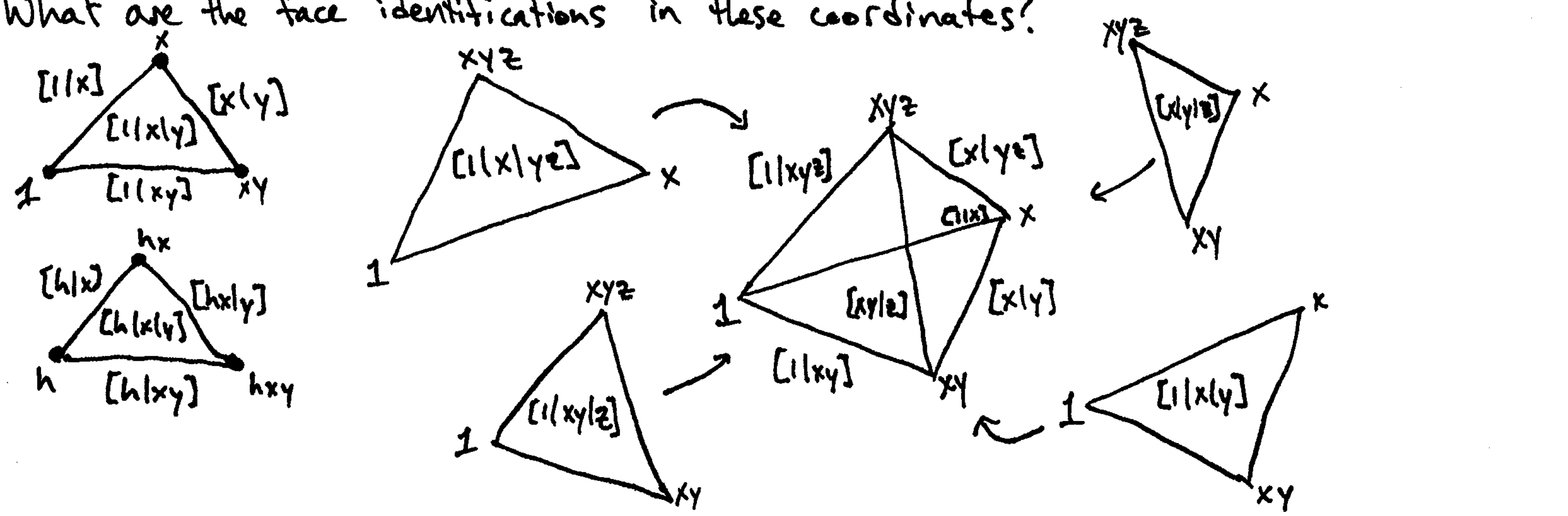
Different coordinates on EG:
label simplices by <u>displacement</u>, not absolute position (also need basepoint)



In general,
$[g_0|g_1|\cdots|g_n] = (g_0, g_0g_1, \ldots, g_0g_1\cdots g_n)$
Action of G:
$g \cdot [g_0|g_1|\cdots|g_n] = [g\cdot g_0|g_1|\cdots|g_n]$

What are the face identifications in these coordinates?



The faces of $[g_0|g_1|\cdots|g_n]$ are $[g_1|\cdots|g_n]$, $[g_0g_1|\cdots|g_n]$, $[g_0|g_1g_2|\cdots|g_n]$, $[g_0|g_1|\cdots|g_{n-1}g_n]$, $[g_0|g_1|\cdots|g_{n-1}]$.

In BG, simplices labeled by $[g_1|\cdots|g_n]$, with faces $[g_1g_2|\cdots|g_n]$, $\ldots$, $[g_1|\cdots|g_{n-1}g_n]$, $[g_1|\cdots|g_{n-1}]$.

Constructing BG when G is a topological group

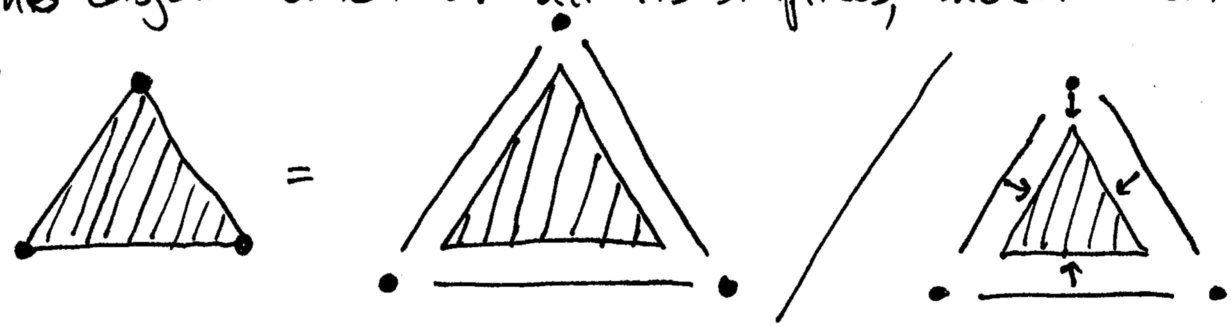
Problem with previous construction: action of G on EG not continuous if G is not discrete.

To fix this, we want to modify the topology on EG so action is continuous.
This can be made into a definition ("the discrete-est topology on EG s.t. the action $G \circlearrowright EG$ is continuous"), but we will need an explicit construction. Fortunately, the answer is very nice.

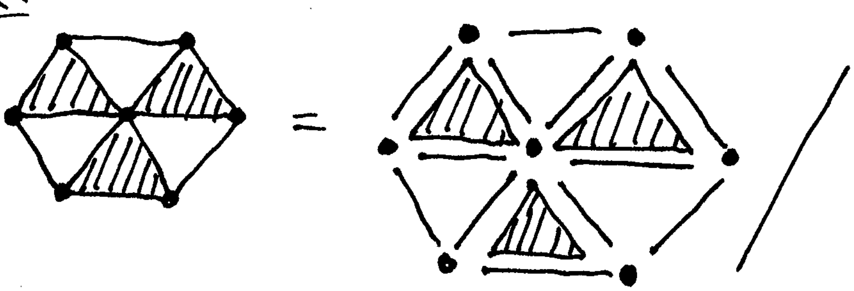
Return briefly to G discrete.

One way to think of a simplicial complex is as the disjoint union of all its simplices, modulo identifications

Ex:



Ex:



identifications

Since n-simplices in BG are labeled by $[g_1|\cdots|g_n] \in \underbrace{G\times\cdots\times G}_{n \text{ times}}$, all n-simplices together look like $G\times\cdots\times G\times\Delta^n$.

$$BG = \coprod_{n\geq 0} G\times\cdots\times G\times\Delta^n \;/\; \text{identifications}$$

Key observation: this definition makes sense for *any* topological group G.

$$EG = \coprod_{n\geq 0} G\times G\times\cdots\times G\times\Delta^n \;/\; \text{identifications}$$

$$BG = \coprod_{n\geq 0} G\times\cdots\times G\times\Delta^n \;/\; \text{identifications} = EG/G$$

Comparing G with ΩBG

There is a natural map $G \to \Omega BG$:
send $g \in G$ to the edge $1 \xrightarrow{[1|g]} g$ in EG, which becomes a loop $[g]$ in BG.

Since $\begin{matrix} G \to EG \\ \downarrow \\ BG \end{matrix}$ is a fiber bundle, we get a long exact sequence in homotopy:

$$\to \underset{\overset{\|}{0}}{\pi_2(EG)} \to \pi_2(BG) \xrightarrow[\cong]{} \pi_1(G) \to \underset{\overset{\|}{0}}{\pi_1(EG)} \to \pi_1(BG) \xrightarrow[\cong]{} \pi_0(G) \to \underset{\overset{\|}{0}}{\pi_0(EG)}$$

So $\pi_{n+1}(BG) \cong \pi_n(G)$ for all $n \geq 0$.

But $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ for any X (just curry your function $S^{n+1} \to X$
as $S^n \to [S^1 \to X]$)

So $\pi_n(\Omega BG) \cong \pi_{n+1}(BG) \cong \pi_n(G)$, and it's easy to see this is induced by $G \to \Omega BG$.
So by Whitehead's Theorem, $G \simeq \Omega BG$.

Final remark: Why is it that $BS^1 = \mathbb{C}P^\infty$, $B\mathbb{C}^* = \mathbb{C}P^\infty$, and $BGL_2^+\mathbb{R} = \mathbb{C}P^\infty$ all coincide?
(It's not just that $S^1 \simeq \mathbb{C}^* \simeq GL_2^+\mathbb{R}$.)

Key: there is a homomorphism $S^1 \to \mathbb{C}^*$ which is a homotopy equivalence.

In general, if a homomorphism $f: G \to H$ is a homotopy equivalence, then $BG \simeq BH$.

- Any homomorphism $f: G \to H$ induces a map $F: BG \to BH$.
- Two ways to finish proof:

I)
$$\begin{array}{ccc} \pi_n(BG) & \xrightarrow[F_*]{} & \pi_n(BH) \\ \| & & \| \\ \pi_{n-1}(G) & \xrightarrow[f_*]{} & \pi_{n-1}(H) \end{array}$$
If all f_* are isomorphisms, so are all F_*; now apply Whitehead's Theorem.

II) We can construct the induced map $F: BG \to BH$ as follows:
the map $f: G \to H$ induces maps $F_n: G \times \cdots \times G \times \Delta^n \to H \times \cdots \times H \times \Delta^n$,
and the fact that f is a homomorphism means the maps F_n respect the identifications. Thus the F_n glue together to give

$$F: BG = \coprod_{n \geq 0} G \times \cdots \times G \times \Delta^n /\sim \longrightarrow \coprod_{n \geq 0} H \times \cdots \times H \times \Delta^n /\sim = BH.$$

If $f: G \to H$ is a homotopy equivalence, then
$F_n: G \times \cdots \times G \times \Delta^n \to H \times \cdots \times H \times \Delta^n$ is a homotopy equivalence for all n.

It can be shown that this implies that
$F: BG \to BH$ is a homotopy equivalence.

(This proof will apply also to homomorphisms of topological monoids.)